Switching methods for the construction of cospectral graphs

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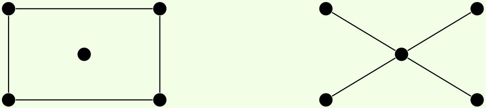
Joint work with Aida Abiad (TU/e) and Nils van de Berg (TU/e)

Cospectral graphs



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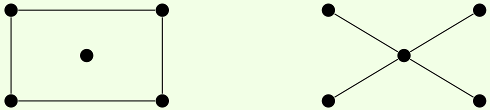
Cospectral graphs



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Definition

Graphs with the same spectrum are **cospectral**.

Cospectral graphs: conjecture

Conjecture (van Dam and Haemers, 2003)

Almost all graphs are determined by their spectrum.

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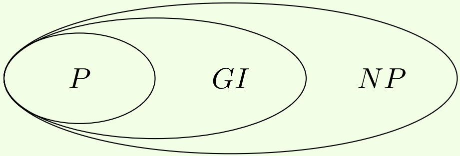

Figure: Is graph isomorphism an easy or hard problem?

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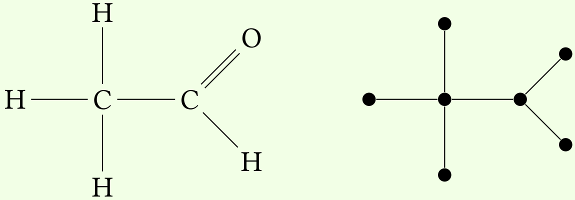

Figure: The molecular graph of acetaldehyde (ethanal).

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🙂 Computational evidence [Brouwer and Spence, 2009]

n	3	4	5	6	7	8	9	10	11
ratio	1	1	0.941	0.936	0.895	0.861	0.814	0.787	0.789

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🙂 Exponentially many graphs are determined by their spectrum [Koval and Kwan, 2023]

$\mathbb{R}$-cospectral graphs

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Two graphs with adjacency matrices A and A' such that

$$A + rJ \quad \text{and} \quad A' + rJ$$

have the same spectrum for all $r \in \mathbb{R}$, are $\mathbb{R}$-**cospectral**.

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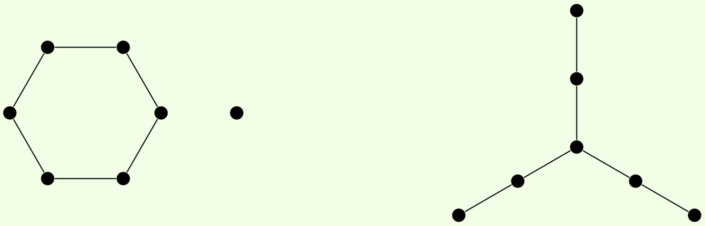

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Let Γ and Γ' be two graphs with adjacency matrices A and A'. The following are equivalent:

- *Γ and Γ' are $\mathbb{R}$-cospectral.*
- *Γ and Γ' are cospectral and so are their complements.*
- *There is a **regular orthogonal** matrix Q such that $A' = Q^T AQ$.*

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$\implies r = \pm 1$ w.l.o.g. $\boxed{r = 1}$

$\mathbb{R}$-cospectral graphs: weaker conjecture

Conjecture

Almost all graphs are determined by their spectrum and the spectrum of their complement.

$\mathbb{R}$-cospectral graphs: weaker conjecture

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Almost all graphs are determined by their spectrum and the spectrum of their complement.

🙂 Sufficient condition for a graph to be determined by its spectrum and the spectrum of its complement [Wang and Xu, 2006]

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level $1 \iff Q$ is a permutation matrix

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level $2 \iff ?$

Regular orthogonal matrices of level 2

Theorem (Chan, Rodger and Seberry, 1986)

Up to permutations of rows and columns, an indecomposable regular orthogonal matrix of level 2 *and row sum* 1 *is one of the following:*

$$(i)\ \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},\quad (ii)\ \frac{1}{2}\begin{bmatrix} J & O & \cdots & \cdots & O & Y \\ Y & J & O & \cdots & \cdots & O \\ O & Y & J & O & \cdots & O \\ & \ddots & \ddots & \ddots & \ddots & \\ O & \cdots & O & Y & J & O \\ O & \cdots & \cdots & O & Y & J \end{bmatrix},$$

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where I, J, O, $Y = 2I - J$ *and* $Z = J - I$, *are* 2×2 *matrices.*

Previous work on level 2

- A. Abiad and W.H. Haemers, Cospectral Graphs and Regular Orthogonal Matrices of Level 2. *Electron. J. Comb.* **19** (2012), P13.

- L. Mao, W. Wang, F. Liu and L. Qiu, Constructing cospectral graphs via regular rational orthogonal matrices with level two. *Discrete Math.* **346** (2023), 113156.

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Definition

A **switching method** is a graph operation, resulting in a cospectral graph. It needs a *switching set* with some conditions.

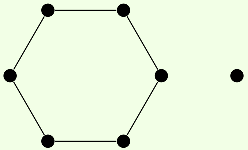
Known switching methods: GM-switching

Theorem (Godsil and McKay, 1982)

Let Γ be a graph with a subgraph C such that:

- *C is regular.*
- *Every vertex outside C has 0, $\frac{1}{2}|C|$ or $|C|$ neighbours in C.*

For every $v \notin C$ that has exactly $\frac{1}{2}|C|$ neighbours in C, reverse its adjacencies with C. The resulting graph is $\mathbb{R}$-cospectral with Γ.



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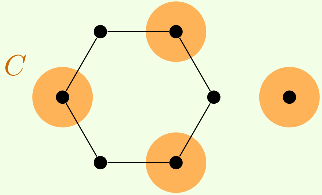

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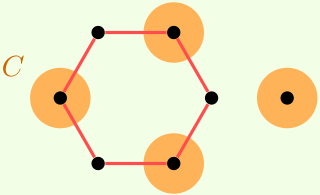

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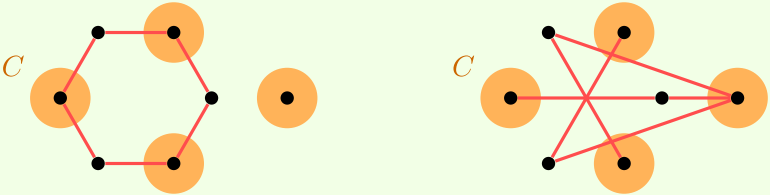

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Proof.

$$\begin{pmatrix} A_{11} & A'_{12} \\ A'_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \frac{2}{|C|}J - I & O \\ O & I \end{pmatrix}^T \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \frac{2}{|C|}J - I & O \\ O & I \end{pmatrix}.$$

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Theorem (Wang, Qiu and Hu, 2019)

Let Γ be a graph with disjoint subgraphs C_1, C_2 such that:

- $|C_1| = |C_2|$.
- *There is a constant c such that, for every vertex of C_i, the number of neighbours in C_i minus the number of neighbours in C_j, is c.*
- *Every vertex outside $C_1 \cup C_2$ has either:*
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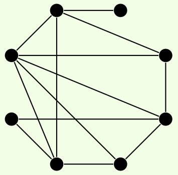

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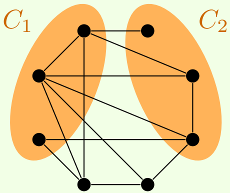

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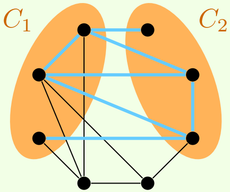

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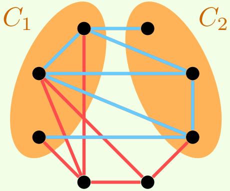

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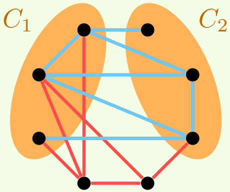

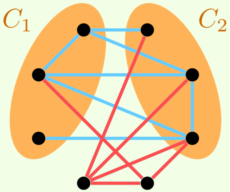

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Up to permutations of rows and columns, an indecomposable regular orthogonal matrix of level 2 and row sum 1 is one of the following:

GM/WQH

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Level 2 switching methods

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AH

new methods

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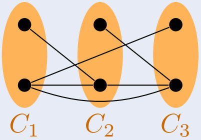
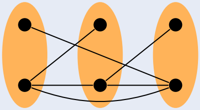
Known switching methods: AH-switching

Theorem (Abiad and Haemers, 2012)

Let Γ be a graph with a vertex partition $\{C_1, C_2, C_3, D\}$ such that:

- *The induced subgraph on $C_1 \cup C_2 \cup C_3$ is*  .
- *Every vertex in D has the same number of neighbours in C_1, C_2 and C_3 mod 2.*

Let π be the permutation on $C_1 \cup C_2 \cup C_3$ that shifts the vertices cyclically to the right. For every $v \in D$ that has exactly one neighbour w in each C_i, replace each edge $\{v, w\}$ by $\{v, \pi(w)\}$.

Replace the induced subgraph on $C_1 \cup C_2 \cup C_3$ by  .

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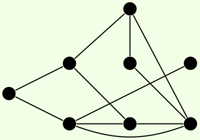

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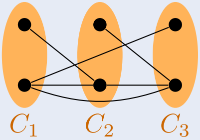
- *The induced subgraph on $C_1 \cup C_2 \cup C_3$ is*  *.*

- *Every vertex in D has the same number of neighbours in C_1, C_2 and C_3 mod 2.*

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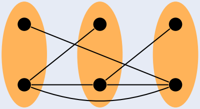
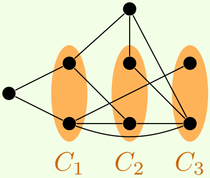
Replace the induced subgraph on $C_1 \cup C_2 \cup C_3$ by  *.*

The resulting graph is $\mathbb{R}$-cospectral with Γ.



Known switching methods: AH-switching

Theorem (Abiad and Haemers, 2012)

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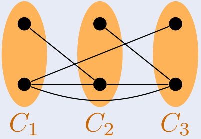
- *The induced subgraph on $C_1 \cup C_2 \cup C_3$ is*  .

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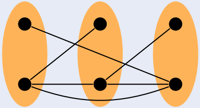
Replace the induced subgraph on $C_1 \cup C_2 \cup C_3$ by  .

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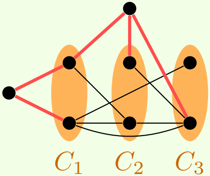

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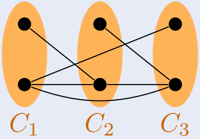
- *The induced subgraph on $C_1 \cup C_2 \cup C_3$ is*  .

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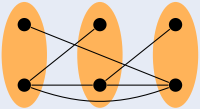
Replace the induced subgraph on $C_1 \cup C_2 \cup C_3$ by  .

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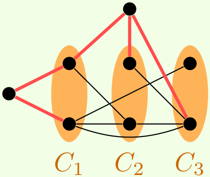
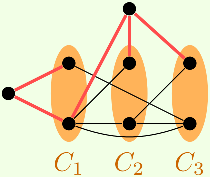




New switching methods: 8 vertices

Proposition (Abiad, van de Berg and Simoens, 2024+)

Every switching that corresponds to a conjugation with the matrix

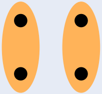
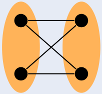
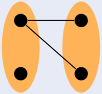
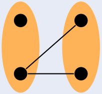
$$\frac{1}{2}\begin{bmatrix} J & O & O & Y \\ Y & J & O & O \\ O & Y & J & O \\ O & O & Y & J \end{bmatrix}$$

can be obtained by repeated GM- and AH-switching.

New switching methods: 10 vertices

Theorem (Abiad, van de Berg and Simoens, 2024+)

Let Γ be a graph with a vertex partition $\{C_1, C_2, C_3, C_4, C_5, D\}$ such that:

- $|C_1| = |C_2| = |C_3| = |C_4| = |C_5| = 2$.
- *Every vertex in D has the same number of neighbours in C_1, C_2, C_3, C_4 and C_5 modulo 2.*
- *For all $i \in \mathbb{Z}/5\mathbb{Z}$, the induced subgraph on $C_i \cup C_{i+1}$ is*  *or*  *and the induced subgraph on $C_i \cup C_{i+2}$ is*  *or* 

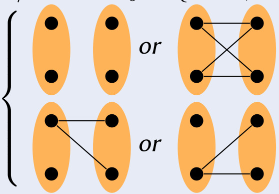
Let π be the permutation on $C_1 \cup \cdots \cup C_5$ that shifts the vertices cyclically to the right. For every $v \notin C$ that has exactly one neighbour w in each C_i, replace each edge $\{v, w\}$ by $\{v, \pi(w)\}$.
For all $i \in \mathbb{Z}/5\mathbb{Z}$, replace the induced subgraph on $C_i \cup C_{i+2}$ by the former induced subgraph on $C_{i-1} \cup C_{i+2}$.
The resulting graph is $\mathbb{R}$-cospectral with Γ.

New switching methods: $2m$ vertices

Theorem (Abiad, van de Berg and Simoens, 2024+)

Let Γ be a graph with a vertex partition $\{C_1, \cdots, C_m, D\}$, m odd, such that:

- $|C_1| = \cdots = |C_m| = 2$.
- *Every vertex in D has the same number of neighbours in C_1, C_2, C_3, C_4 and C_5 modulo 2.*
- *For all $i, j \in \mathbb{Z}/m\mathbb{Z}$ with $j \in \{i+1, \ldots, i + \frac{m-1}{2}\}$, $C_i \cup C_j$ is*



if $j < i + \frac{m-1}{2}$

if $j = i + \frac{m-1}{2}$

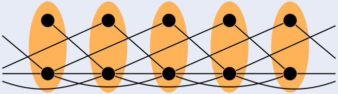
Let π be the permutation on $C_1 \cup \cdots \cup C_m$ that shifts the vertices cyclically to the right. For every $v \notin C$ that has exactly one neighbour w in each C_i, replace each edge $\{v, w\}$ by $\{v, \pi(w)\}$.
For all $i \in \mathbb{Z}/m\mathbb{Z}$, replace the induced subgraph on $C_i \cup C_{i+\frac{m-1}{2}}$ by the former induced subgraph on $C_{i-1} \cup C_{i+\frac{m-1}{2}}$.
The resulting graph is $\mathbb{R}$-cospectral with Γ.

New switching methods: $2m$ vertices

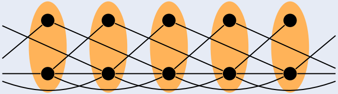
Theorem (Abiad, van de Berg and Simoens, 2024+)

Let Γ be a graph with a vertex partition $\{C_1, \cdots, C_m, D\}$, $m \geq 4$, such that:

- $|C_1| = \cdots = |C_m| = 2$.
- *Every vertex in D has the same number of neighbours in C_1, C_2, C_3, C_4 and C_5 modulo 2.*
- *The induced subgraph on $C_1 \cup \cdots \cup C_m$ is*



Let π be the permutation on $C_1 \cup \cdots \cup C_m$ that shifts the vertices cyclically to the right. For every $v \notin C$ that has exactly one neighbour w in each C_i, replace each edge $\{v, w\}$ by $\{v, \pi(w)\}$.
Replace the induced subgraph on $C_1 \cup \cdots \cup C_m$ by



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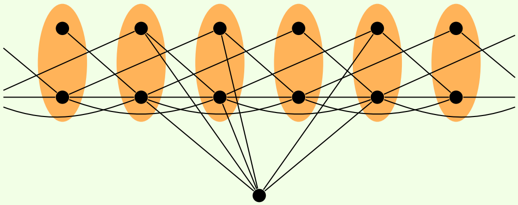
New switching methods: $2m$ vertices



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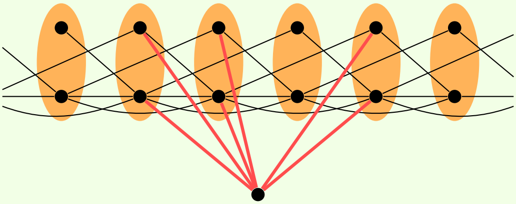

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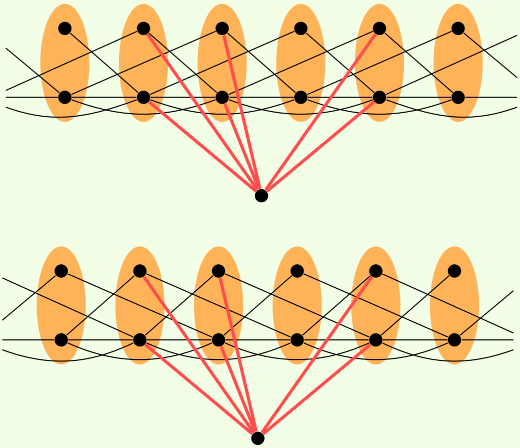

Overview new results

- Classification of level 2 switching methods with blocks up to size 12:
  - 4 vertices: 1 type; GM-/WQH-switching
  - 6 vertices: 1 type; AH-switching
  - 8 vertices: 0 types; always reducible
  - 10 vertices: 3 types
  - 12 vertices: 18 types
- Three new general switching methods with blocks of size $2m$

Asymptotic bounds

Let g_n denote the number of graphs on n vertices.

Theorem (Haemers and Spence, 2004)

At least $n^3 g_{n-1}\left(\frac{1}{24}+o(1)\right)$ graphs on n vertices are not determined by their spectrum.

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There are $n^3 g_{n-1} \left(\frac{1}{24} + o(1)\right)$ graphs on n vertices with a GM-switching set of size 4.

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There are $n^3 g_{n-1}\left(\frac{1}{24}+o(1)\right)$ graphs on n vertices with a GM-switching set of size 4.

Theorem (Abiad, van de Berg and Simoens, 2024+)

There are between

$$n^4 g_{n-2}\left(\frac{1}{72}-o(1)\right) \qquad \textit{and} \qquad n^4 g_{n-2}\left(\frac{11}{8}\right)^{n-6}\left(2^9+o(1)\right)$$

graphs on n vertices with a WQH-switching set of size 6.

Further research

Ongoing work:

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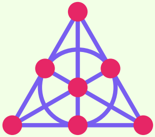
$$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$



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- Asymptotic bounds on Abiad-Haemers switching
- Numerical experiments with new switching methods
- Sporadic 8 vertex switching
- Fano switching

Open problems:

- Classification of regular orthogonal matrices of level 3

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Thank you for listening!